Technical Report No. 32-127

Numerical Determination of Radiation Configuration Factors for Some Common Geometrical Situations

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ABSTRACT

A derivation of the general integral for the determination of thermal-radiation configuration factors is presented, together with a scheme for its solution using JPL's IBM 7090 computer. An analytical method for determining the rather complex analytical expression for the integrand and limits of integration between a particular pair of geometrical shapes is outlined. Two examples of the method's employment are presented. Finally, a table of integrals for configuration factors between pairs of commonly encountered geometrical shapes is given. These integrals, with two exceptions, are completely general in regard to the relative orientation and size of the geometries.

I. STATEMENT OF THE PROBLEM

For spacecraft the primary mode of heat transfer between external surfaces and in many instances between internal surfaces is thermal radiation. Because of the geometrical complexity of spacecraft, the analytical determination of the net heat-exchange rates between surfaces is mathematically very difficult. Exact mathematical solutions are virtually impossible. To make mathematical analysis of radiant heat exchange feasible, the spacecraft is divided into units and each unit is analyzed, assuming its surface to be isothermal. This assumption introduces some error, the magnitude of which depends upon the thermal situation. In general, errors which result from dividing the spacecraft into units and postulating that the surfaces of each unit are isothermal can be minimized by careful subdivision and by dealing with small units.

The type of equation generally used in determining the thermal interactions between isothermal surfaces separated by a non-absorbing medium is[1]

$$q_{12} = \sigma A_1 f_{12}(T_1^4 - T_2^4) \tag{1}$$

where:

q_{12} = net heat-exchange rate between Surface 1 and Surface 2, Btu/hr

σ = Stefan-Boltzman constant = 1.19×10^{-10} Btu/in^{2}hr°R^4

A_1 = area of Surface 1, in^2

f_{12} = an influence coefficient, dimensionless

T_1 = temperature of Surface 1, °R

T_2 = temperature of Surface 2, °R

Even using simplified equations such as Eq. (1) mathematical analysis, though feasible, is still quite difficult. The major obstacle is the determination of the influence coefficient f_{12} which is a function of the emissivities and absorptivities of surfaces A_1 and A_2 and the configuration factor between the surfaces. Though considerable data are available regarding values of emissivity and absorptivity, little is available regarding configuration factors; thus, this is crux of the problem. Without configuration factors, meaningful analysis is impossible. This lack of configuration factor data is to be expected, however, since configuration factors are functions of the geometries and relative orientations of the surfaces involved, of which a great number of variations are of practical interest.

[1]McAdams, W. H., *Heat Transmission*, McGraw-Hill Book Co., Third Edition, 1954.

II. DERIVATION OF THE GENERAL INTEGRAL FOR CONFIGURATION FACTORS

In arriving at a configuration factor, the difficulty arises from the inability to obtain solutions to the following general quartic integral

$$F_{12} = \frac{1}{\pi A_1} \int_{A_1} \int_{A_2} \frac{\cos \phi_1 \cos \phi_2}{r^2} \, dA_1 \, dA_2 \tag{2}$$

where:

F_{12} = the configuration factor

A_1 = the area of surface A_1 as viewed from surface A_2

A_2 = the area of surface A_2 as viewed from dA_1

r = the magnitude of the vector **r** between the elemental areas dA_1 and dA_2

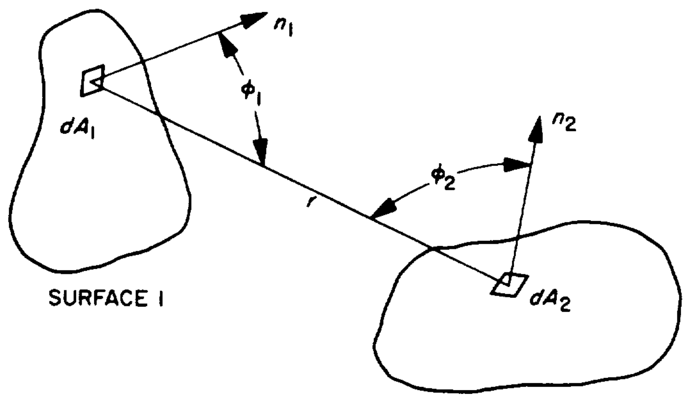
ϕ_1 = the angle formed by **r** and the normal to dA_1

ϕ_2 = the angle formed by **r** and the normal to dA_2

For curved surfaces, the limits of the inner integral are, in general, functions of the location of the elemental area, dA_1. For a particular pair of surfaces and their relative orientation, the integral becomes very difficult indeed to integrate, when the proper expressions are substituted for the terms appearing in the integrand and limits.

To clarify the meaning of the integral, a derivation of Eq. (2) will be presented. The configuration factor between Surfaces 1 and 2 is defined as that fraction of the total flux leaving Surface 1 incident on Surface 2. For diffusely radiating surfaces, the configuration factor is a function of the geometries and relative orientation of the surfaces. If the radiation is non-diffuse, the configuration factor is also a function of the directional distribution of the flux leaving Surface 1.

The sketch following shows an elemental area dA_1 of total emissive power W, radiating in all directions from one side with some of its radiation being intercepted by another elemental area dA_2 at a distance r from dA_1. A line connecting the elemental areas dA_1 and dA_2 forms the angles ϕ_1 and ϕ_2 with the



normals n_1 and n_2 to the elemental areas, respectively. The heat-transfer rate from dA_1 to dA_2, written as $dq_{1\to 2}$, is directly proportional to the product of the apparent area of dA_1 as seen from dA_2 (or $dA_1 \cos \phi_1$) and the apparent area of dA_2 as seen from dA_1 or ($dA_2 \cos \phi_2$) and also is inversely proportional to the square of the separation distance r. A proportionality term I_1 which is the emitted flux at the surface of dA_1, must be introduced. Thus, $dq_{1\to 2}$ is

$$dq_{1\to 2} = \frac{I_1\, dA_1 \cos \phi_1\, dA_2 \cos \phi_2}{r^2} \tag{3}$$

If the flux leaving Surface 1 is not diffuse, I_1 is a function of the angle ϕ_1; thus, Eq. (3) becomes

$$dq_{1\to 2} = \frac{I_1(\phi_1) \cos \phi_1 \cos \phi_2\, dA_1\, dA_2}{r^2} \tag{4}$$

Since W is the total emissive power of Surface 1, then $dq_{1\to 2}$ can also be expressed by

$$dq_{1\to 2} = W dA_1 F_{dA_1 \to dA_2} \tag{5}$$

where $F_{dA_1 \to dA_2}$ is the configuration factor between the elemental areas dA_1 and dA_2. Setting Eq. (4) and (5) equal yields

$$dA_1 F_{dA_1 \to dA_2} = \frac{I_1(\phi_1) \cos \phi_1 \cos \phi_2\, dA_1\, dA_2}{W r^2} \tag{6}$$

Now, W is related to the mean directional flux I_m leaving dA_1 by $W = \pi I_m$; thus Eq. (6) becomes

$$dA_1 \, F_{dA_1 \to dA_2} = \frac{I_1(\phi_1)}{\pi I_m} \; \frac{\cos \phi_1 \cos \phi_2 \, dA_1 \, dA_2}{r^2} \tag{7}$$

To determine the configuration factor between Surfaces 1 and 2, integrate first dA_2 over A_2 then integrate dA_1 over A_1. The first integration yields

$$dA_1 \, F_{dA_1 \to A_2} = dA_1 \int_{A_2} \frac{I_1(\phi_1)}{\pi I_m} \; \frac{\cos \phi_1 \cos \phi_2 \, dA_2}{r^2} \tag{8}$$

and the second integration yields

$$A_1 \, F_{A_1 \to A_2} = A_1 \, F_{12} = \int_{A_1} \int_{A_2} \frac{I_1(\phi_1)}{\pi I_m} \; \frac{\cos \phi_1 \cos \phi_2 \, dA_1 \, dA_2}{r^2} \tag{9}$$

It will be assumed from this point that Surfaces 1 and 2 are diffusely radiating and, therefore, $I_1(\phi_1)/I_m = 1$ for all angles of ϕ_1. Thus Eq. (6) becomes

$$F_{12} = \frac{1}{\pi A_1} \int_{A_1} \int_{A_2} \frac{\cos \phi_1 \cos \phi_2 \, dA_1 \, dA_2}{r^2} \tag{10}$$

It should be pointed out that for diffusely radiating surfaces, the reciprocity principle is valid; that is,

$$A_1 \, F_{12} \equiv A_2 \, F_{21} \tag{11}$$

Until recently, solutions of Eq. (10) appearing in the literature for geometrically simple pairs of surfaces of very simple relative orientation (e.g., perpendicular rectangles sharing a common edge), were sufficient to make the necessary calculations. However, these solutions can no longer be applied to current problems because of their geometrical complexity. Therefore, it is imperative that a suitable technique be devised for obtaining solutions of the integral. Numerical integration using a digital computer was found to be suitable and desirable, since it permits parametric studies of various surfaces to be made and yet is flexible, convenient, and rapid.

The technique relies upon the fact that complex surfaces can, with a few exceptions, be broken down or divided into units of elementary shapes which can be represented by analytical expressions, such as cones, cylinders, plates, and spheres. The technique is essentially as follows: Equation (10) is developed for a given combination of elementary shapes in a general manner so that the derived integral is applicable to any relative orientation between the shapes, with as few exceptions as possible. The resulting integral is then programmed for the computer. Various constants and limits of integration which define a particular geometrical situation are supplied by the engineer for each specific case. Clearly, once a general integral has been developed for a particular combination of surfaces, the integral can be repeatedly used for different orientations and physical sizes, simply by changing the constants and limits of integration. Since it is necessary to develop a separate integral for each combination of geometries, representative geometries were studied to determine the degree of difficulty involved in developing an integral. It was found that the derivation of the integrand and limits could be expeditiously carried out and that the derivation is rather straightforward.

III. PROCEDURE FOR DETERMINING INTEGRAND AND LIMITS OF INTEGRATION

Let P_1 and P_2, P_1 and P_3, and P_2 and P_4 be two points respectively on each of the lines, r, n_1 and n_2 (see sketch below). The analytical expression for the separation distance r is simply the expression for the length of a line, or

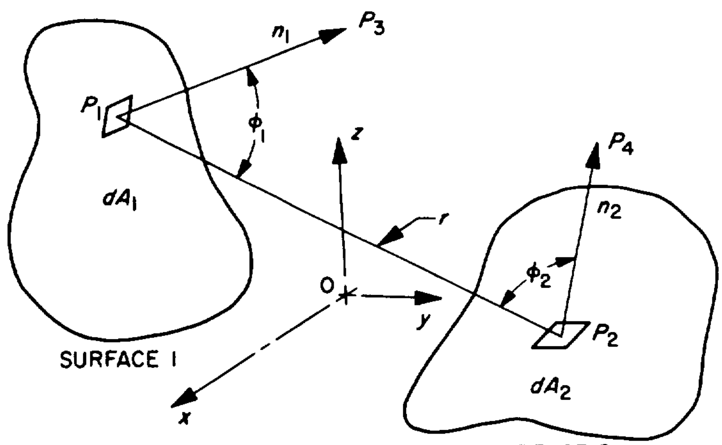
$$r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \tag{12}$$

where the points P_1 and P_2 lie at the centroids of the elemental areas dA_1 and dA_2. The lengths of the normals between the points are

$$n_1^2 = (x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2 \tag{13}$$

and

$$n_2^2 = (x_2 - x_4)^2 + (y_2 - y_4)^2 + (z_2 - z_4)^2 \tag{14}$$



Recall that the angles ϕ_1 and ϕ_2 are formed by r and the normals n_1 and n_2 respectively. If α, β, and γ are the direction angles of r; α_1, β_1, and γ_1 are the direction angles of n_1; and α_2, β_2, and γ_2 are the direction angles of n_2; then

$$\cos \phi_1 = \cos \alpha \cos \alpha_1 + \cos \beta \cos \beta_1 + \cos \gamma \cos \gamma_1 \tag{15}$$

and

$$\cos \phi_2 = \cos \alpha \cos \alpha_2 + \cos \beta \cos \beta_2 + \cos \gamma \cos \gamma_2 \tag{16}$$

Using the points on and the lengths of r_1, n_1, and n_2, the direction cosines of these lines are obtained from

$$\left.\begin{array}{lll} \cos \alpha = \dfrac{x_1 - x_2}{r}, & \cos \alpha_1 = \dfrac{x_3 - x_1}{n_1}, & \cos \gamma_2 = \dfrac{x_2 - x_4}{n_2} \\[2ex] \cos \beta = \dfrac{y_1 - y_2}{r}, & \cos \beta_1 = \dfrac{y_3 - y_1}{n_1}, & \cos \beta_2 = \dfrac{y_2 - y_4}{n_2} \\[2ex] \cos \gamma = \dfrac{z_1 - z_2}{r}, & \cos \gamma_1 = \dfrac{z_3 - z_1}{n_1}, & \cos \gamma_2 = \dfrac{z_2 - z_4}{n_2} \end{array}\right\} \tag{17}$$

Thus,

$$\cos \phi_1 = \frac{x_1 - x_2}{r} \frac{x_3 - x_1}{n_1} + \frac{y_1 - y_2}{r} \frac{y_3 - y_1}{n_1} + \frac{z_1 - z_2}{r} \frac{z_3 - z_1}{n_1} \tag{18}$$

and

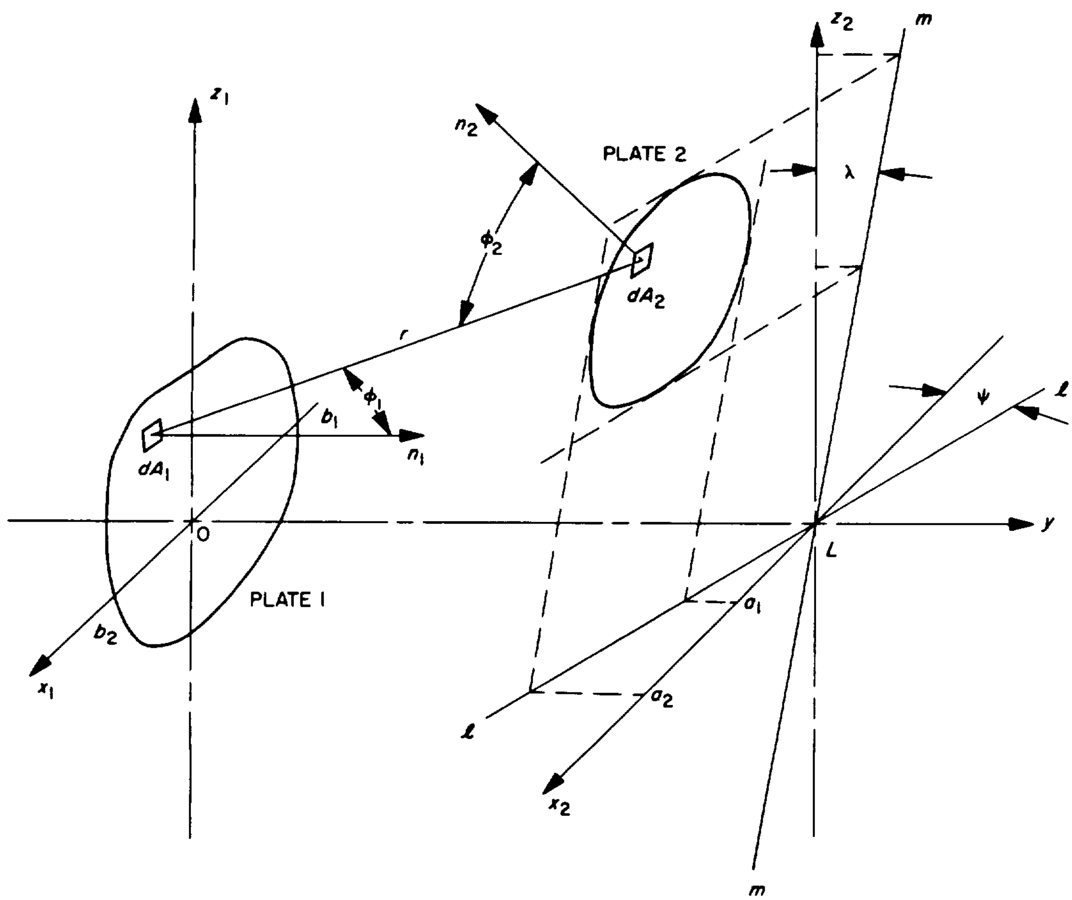
$$\cos \phi_2 = \frac{x_1 - x_2}{r} \frac{x_2 - x_4}{n_2} + \frac{y_1 - y_2}{r} \frac{y_2 - y_4}{n_2} + \frac{z_1 - z_2}{r} \frac{z_2 - z_4}{n_2} \tag{19}$$

When curved surfaces are involved, the area viewed of either or both of the Surfaces 1 and 2 may be functions, respectively, of the locations of dA_1 and dA_2. For example, that area A_1 of a sphere, which is in view of an elemental area dA_2, depends upon the location of dA_2. In such situations, the limits of integration (that is, boundaries of the viewed area), are obtained by setting either or both of the functions $\cos \phi_1$ and $\cos \phi_2$ equal to zero.

Recapitulating, the procedure is: (1) determine two points on each of the lines r_1, n_1, and n_2; (2) determine the lengths between the points; (3) using the points and lengths, determine the direction cosines of the lines r_1, n_1, and n_2; and (4) using the direction cosines, determine the analytical expression for cos ϕ_1 and cos ϕ_2.

A. Example: Two Arbitrarily Oriented Plates

As an example, consider the very common situation of two flat plates of arbitrary size and relative orientation, as shown below. The coordinate system was chosen so that Surface 1 lies in the $y_1 = 0$ plane.[2]



The line l lies both in the plane of Surface 2 and in the $z_2 = 0$ plane; thus, line l represents the line formed by the intersection of the plane of Surface 2 and the x_2y_2 plane. The line m lies both in the plane of Surface 2 and in the $x_2 = 0$ plane; thus, it represents the line formed by the intersection of the plane of plate 2 and the y_2z_2 plane. Surface 2 lies in the plane defined by the coplanar lines l and m. The x_2z_2 coordinate axes originate at the intersection of l and m with the y axis which occurs at a distance L along y. These axes are parallel and coplanar with the x_1z_1 coordinate axes. Two angles are formed, ψ and λ, which describe the orientation of Surface 2 to Surface 1. With the coordinates defined, an integrand for Eq. (10) can now be derived.

[2] Coordinates y and y_1 are identical.

Using the centroids of the elemental areas, dA_1 and dA_2, as the locations of two points on r (a necessary selection) yields

$$(x_1, 0, z_1)$$

and

$$(x_2, L - x_2 \tan\psi + z_2 \tan\lambda, z_2)$$

where $L - x_2 \tan\psi + z_2 \tan\lambda$ is the y point of dA_2.

From Eq. (13), the separation distance between dA_1 and dA_2 is

$$r^2 = (x_2 - x_1)^2 + (L - x_2 \tan\psi + x_2 \tan\lambda)^2 + (z_2 - z_1)^2 \tag{20}$$

The direction cosines of r are

$$\left.\begin{aligned} \cos\alpha &= \frac{x_2 - x_1}{r} \\ \cos\beta &= \frac{L - x_2 \tan\psi + z_2 \tan\lambda}{r} \\ \cos\gamma &= \frac{z_2 - z_1}{r} \end{aligned}\right\} \tag{21}$$

Since n_1 originates at the centroid of dA_1, one point on n_1 is

$$(x_1, 0, z_1)$$

For the other point, it is convenient to use the intersection of n_1 with the plane of Surface 2; thus,

$$(x_1, L - x_1 \tan\psi + z_1 \tan\lambda, z_1)$$

The distance between these points is

$$n_1 = (x_1 - x_1)^2 + (L - x_1 \tan\psi + z_1 \tan\lambda)^2 + (z_1 - z_1)^2$$

or

$$n_1 = L - x_1 \tan\psi + z_1 \tan\lambda \tag{22}$$

The direction cosines are

$$\left.\begin{aligned}
\cos\alpha_1 &= \frac{x_1 - x_1}{n_1} \\
\cos\beta_1 &= \frac{L - x_1 \tan\psi - z_1 \tan\lambda}{n} \\
\cos\gamma_1 &= \frac{z_1 - z_1}{r}
\end{aligned}\right\} \tag{23}$$

or $\cos\alpha_1 = 0$, $\cos\beta_1 = 1$, and $\cos\gamma_1 = 0$. Then from the direction cosines of n_1 and r,

$$\cos\phi_1 = \frac{L - x_2 \tan\psi + z_2 \tan\lambda}{r} \tag{24}$$

Using the centroidal point of dA_2 as one point on n_2, and the intersection point of n_2 with the plane of Plate 1 as the other:

$$(x_2,\ L - x_2 \tan\psi + z_2 \tan\lambda,\ z_2)$$

and

$$[x_2 - (L - x_2 \tan\psi + z_2 \tan\lambda) \tan\psi,\ 0,\ z_2 + (L - x_2 \tan\psi + z_2 \tan\lambda) \tan\lambda]$$

the length of n_2 between these two points is

$$n_2^2 = [x_2 - x_2 + (L - x_2 \tan\psi + z_2 \tan\lambda)\tan\psi]^2 + (L - x_2 \tan\psi + z_2 \tan\lambda)^2$$
$$+ [z_2 - z_2 - (L - x_2 \tan\psi + z_2 \tan\lambda) \tan\lambda]^2$$

or

$$n_2 = (L - x_2 \tan\psi + z_2 \tan\lambda)\,(1 + \tan^2\psi + \tan^2\lambda)^{\frac{1}{2}} \tag{25}$$

The direction cosines of n_2 are

$$\left.\begin{aligned} \cos\alpha_2 &= \frac{\tan\psi}{(1 + \tan^2\psi + \tan^2\lambda)^{\frac{1}{2}}} \\ \cos\beta_2 &= \frac{1}{(1 + \tan^2\psi + \tan^2\lambda)^{\frac{1}{2}}} \\ \cos\gamma_2 &= -\frac{\tan\lambda}{(1 + \tan^2\psi + \tan^2\lambda)^{\frac{1}{2}}} \end{aligned}\right\} \tag{26}$$

Therefore, $\cos\phi_2$ is

$$\cos\phi_2 = \frac{1}{r\,(1 + \tan^2\psi + \tan^2\lambda)^{\frac{1}{2}}}\left[(x_2 - x_1)\tan\psi + L - x_2\tan\psi + z_2\tan\lambda + (z_2 - z_1)\tan\lambda\right]$$

or

$$\cos\phi_2 = \frac{L - x_1\tan\psi + z_1\tan\lambda}{r(1 + \tan^2\psi + \tan^2\lambda)^{\frac{1}{2}}} \tag{27}$$

Since Surface 1 was located in the $y = 0$ plane, dA_1 is given by the simple expression,

$$dA_1 = dx_1\,dz_1 \tag{28}$$

Since all expressions are in terms of x_1, x_2, z_1, and z_2, the element dA_2 must be expressed in terms of x_2 and z_2. The transposition of dA_2 into the $x_2 z_2$ plane gives the proper expression for dA_2 or

$$dA_2 = \frac{dx_2}{\cos\psi}\,\frac{dz_2}{\cos\lambda} \tag{29}$$

Substituting the preceding expressions into Eq. (10) gives, for the configuration factor between two skewed plates, Eq. (30):

$$F_{12} = \frac{1}{\pi A_1} \int_{A_1} \int_{A_2} \frac{(L - x_1 \tan\psi + z_1 \tan\lambda)(L - x_2 \tan\psi + z_2 \tan\lambda)\, dx_1\, dz_1\, dx_2\, dz_2}{(1 + \tan^2\psi + \tan^2\lambda)^{½} \left[(x_2 - x_1)^2 + (L - x_2 \tan\psi + z_2 \tan\lambda)^2 + (z_2 - z_1)^2\right]^2 \cos\psi \cos\lambda} \tag{30}$$

By programming this integrand for a computer, any configuration factor between two plates with one exception can be determined by supplying to this integral the limits of integration which define the boundaries of the plates and by supplying the constants L, ψ, and λ. Examination of this integral reveals the exception: the integral is not valid as either or both ψ and λ approach 90 deg.

In arriving at the limits of integration, it is assumed that the boundaries of the plate will be mathematically expressible, either exactly or to reasonable accuracy. Therefore, when the integration is carried out along z axes first, the boundaries of Surface 1 and Surface 2 are given by the functional expressions

$$z_1 = f(x_1)$$

and

$$z_2 = g(x_2)$$

Substituting the limits of integration into Eq. (30) yields

$$F_{12} = \frac{1}{\pi A_1} \int_{b_1}^{b_2} \int_{f_1(x_1)}^{f_2(x_2)} \int_{a_1}^{a_2} \int_{g_1(x_2)}^{g_2(x_2)} \frac{(L - x_1 \tan\psi + z_1 \tan\lambda)(L - x_2 \tan\psi + z_2 \tan\lambda)\, dz_2\, dx_2\, dz_1\, dx_1}{(1 + \tan^2\psi + \tan^2\lambda)^{½} \left[(x_2 - x_1)^2 + (L - x_2 \tan\psi + z_2 \tan\lambda)^2 + (z_2 - z_1)^2\right]^2 \cos\psi \cos\lambda} \tag{31}$$

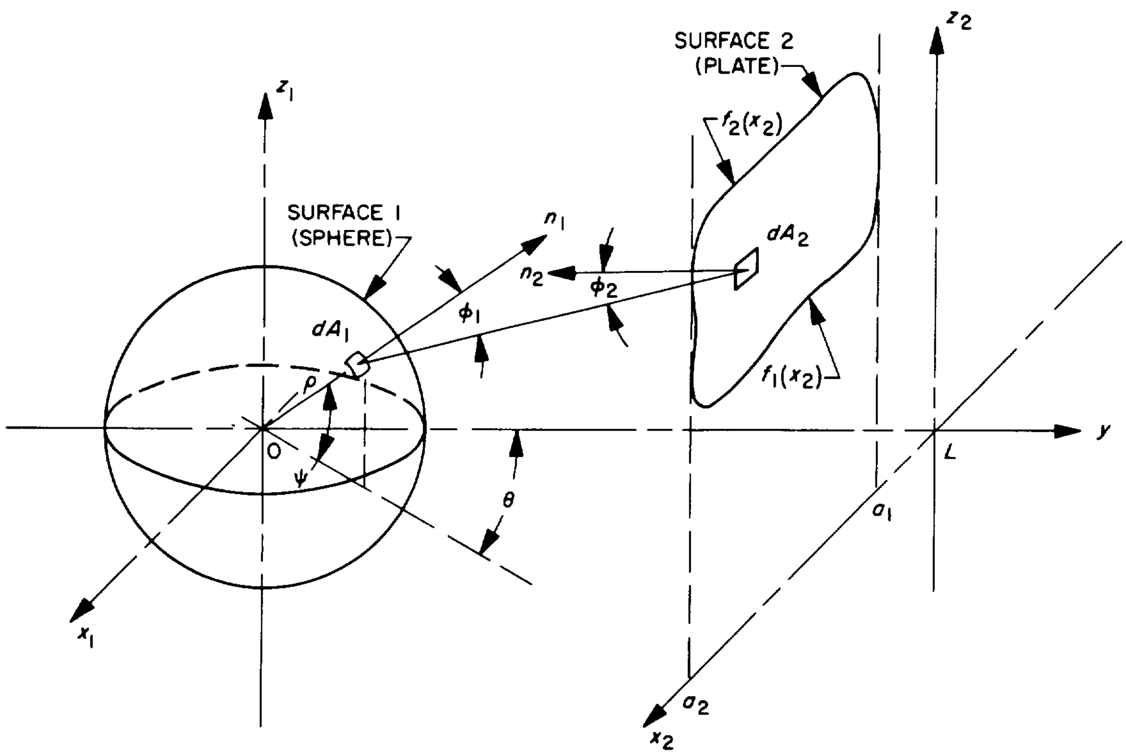
The functions $g_1(x_2)$ and $g_2(x_2)$ must be the expressions of the projections of the boundaries into the $x_2 z_2$ plane. Caution should be exercised in arriving at the limits of integration when Surface 2 is skewed at such an angle that either the back of Surface 2 can be seen from a portion of Surface 1 or vice versa. In such a case, the integration should extend only over that area which is in view of another area with all other areas being treated by a separate integral. The point here is that the integration should always be made only over areas in which dA_1 can see dA_2 or vice versa.

Recall that the coordinate system for the two plates was chosen so that the two arbitrary angles were form ψ and λ. However, two angles are not a necessity, since if the plates are rotated about the y axis while maintaining the same relative orientation, the same degree of generality can be obtained using only one angle. The usefulness of two arbitrary angles becomes apparent by noting that with two angles of skewness, the plates can be placed in a coordinate system arbitrarily with only the one restriction that Surface 1 lies in the $y = 0$ plane; thus, the expressions for the limits of integration are greatly simplified by rotating and/or translating the location of Surface 1 within the $y = 0$ plane. With only one angle of skewness, the plates can be placed into coordinate systems in only one position relative to either the x_1 or the z_1 axis and thus, the limits of integration are fixed. Permitting only one angle of skewness, i.e., $\lambda = 0$, Eq. (31) reduces to the following:

$$F_{12} = \int_{b_1}^{b_2} \int_{f_1(x_1)}^{f_2(x_1)} \int_{a_1}^{a_2} \int_{g_1(x_2)}^{g_2(x_2)} \frac{(L - x_1 \tan \psi)(L - x_2 \tan \psi)}{[(x_2 - x_1)^2 + (L - x_2 \tan \psi)^2 + (z_2 - z_1)^2]^2} \, dz_2 \, dx_2 \, dz_1 \, dx_1 \tag{32}$$

B. Example: Sphere and Plate of Arbitrary Orientation

Consider, as another example, the formulation of Eq. (10) for the configuration factor between a sphere of radius ρ and a plate, as shown in the following sketch. The x_1, y, and z_1 coordinates were chosen so that the origin of the coordinates lies at the center of the sphere. The sphere and plate are then rotated about the center of the sphere until the plate is perpendicular to the $x_1 y$ plane and parallel to the $x_1 z_1$ plane. The $x_2 y_2 z_2$ coordinate system is formed at the intersection of the plane of the plate and the y axis. The x_2 and z_2 axes are parallel and coplanar with the x_1 and z_1 axes, respectively. The origin of the $x_2 y_2 z_2$ system is located at a distance L along the y axis from the origin of the $x_1 y z_1$ system. With the coordinates defined, an integrand for Eq. (10) can now be derived.



Using the centroids of the elemental areas dA_1 and dA_2 as the locations of two points on r, yields

$$(x_1, y, z_1)$$

and

$$(x_2, L, z_2)$$

Transforming x_1, y, and z_1 into spherical coordinates to describe the sphere, yields for (x_1, y, z_1)

$$(\rho \cos \psi \sin \theta,\ \rho \cos \psi \cos \theta,\ \rho \sin \psi)$$

The distance between dA_1 and dA_2 becomes

$$r^2 = (x_2 - \rho \cos \psi \sin \theta)^2 + (L - \rho \cos \psi \cos \theta)^2 + (z_2 - \rho \sin \psi)^2 \tag{33}$$

and the direction cosines of r are

$$\left.\begin{aligned}\cos\alpha &= \frac{x_2 - \rho\cos\psi\sin\theta}{r}\\ \cos\beta &= \frac{L - \rho\cos\psi\cos\theta}{r}\\ \cos\gamma &= \frac{z_2 - \rho\sin\psi}{r}\end{aligned}\right\} \tag{34}$$

Since n_1 passes through the centroid of dA_1, one point on n_1 is $(\rho\cos\psi\sin\theta,\ \rho\cos\psi\cos\theta,\ \rho\sin\psi)$. Because n_1 is normal to the surface of the sphere, it must pass through the origin of the $x_1 y z_1$ coordinate system; hence another point on n_1 is

$$(0, 0, 0)$$

The length of n_1 is, therefore, equal to the radius of the sphere, or $n_1 = \rho$. The direction cosines of n_1 are

$$\left.\begin{aligned}\cos\alpha_1 &= \cos\psi\sin\theta\\ \cos\beta_1 &= \cos\psi\cos\theta\\ \cos\gamma_1 &= \sin\psi\end{aligned}\right\} \tag{35}$$

From the direction cosines of n_1 and r,

$$\cos\phi_1 = \frac{1}{r}\left[(x_2\sin\theta + L\cos\theta)\cos\psi + z_2\sin\psi - \rho\right] \tag{36}$$

Using the centroidal point of dA_2 as one point on n_2 and using the intersection point of dA_2 with the $y = 0$ plane as the other point on n_2 gives

$$(x_2, L, z_2)$$

and

$$(x_2, 0, z_2)$$

The length of n_1 is

$$n_1 = L \tag{37}$$

and the direction cosines are

$$\left.\begin{aligned} \cos \alpha_2 &= 0 \\ \cos \beta_2 &= 1 \\ \cos \gamma_2 &= 1 \end{aligned}\right\} \tag{38}$$

Thus, $\cos \phi_2$ is

$$\cos \phi_2 = \frac{L - \rho \cos \psi \cos \theta}{r} \tag{39}$$

The expressions the elemental areas are

$$\left.\begin{aligned} dA_1 &= \rho^2 d\psi d\theta \\ dA_2 &= dx_2 \, dz_2 \end{aligned}\right\} \tag{40}$$

Substituting the latter into Eq. (10) gives, for the configuration factor, the following integral

$$A_1F_{12} = \frac{\rho^2}{\pi A_1} \int_{A_1} \int_{A_2} \frac{[(x_2 \sin \theta + L \cos \theta) \cos \psi + z_2 \sin \psi - \rho](L - \rho \cos \psi \cos \theta) \, d\psi \, d\theta \, dx \, dz_2}{[(x_2 - \rho \cos \psi \sin \theta)^2 + (L - \rho \cos \psi \cos \theta)^2 + (z_2 - \rho \sin \psi)^2]^2} \tag{41}$$

Since the boundary of the sphere as seen from dA_2 is implicitly defined, it is possible to derive the limits of integration. Obviously, the limits of integration for the sphere are obtained when $\cos \phi_1 = 0$ or $\phi_1 = 90$ deg. Setting $\cos \phi_1 = 0$ yields

$$x_2 \sin\theta \cos\psi + L \cos\theta \cos\psi + z_2 \sin\psi - \rho = 0 \tag{42}$$

Assume that integration is to occur over θ first, then the latter equation must be solved for θ. Letting, for convenience,

$$\left.\begin{aligned} x_2 \cos\psi &= a \\ L \cos\psi &= b \\ \rho - z_2 \sin\psi &= d \end{aligned}\right\} \tag{43}$$

Then

$$a \sin\theta + b \cos\theta = d \tag{44}$$

Now, it can be shown that

$$\theta = \cos^{-1} \frac{d}{(a^2 + b^2)^{1/2}} \pm \tan^{-1} \frac{a}{b} \tag{45}$$

where the minus sign is for the lower limit and the plus sign for the upper limit. Substituting the proper expressions for a, b, and d gives

$$\theta = \cos^{-1} \frac{\rho - z_2 \sin\psi}{(x_2^2 + L^2)^{1/2} \cos\psi} \pm \tan^{-1} \frac{x_2}{L} \tag{46}$$

It can also be shown that the maximum and minimum values of ψ occur when $\cos\phi_1 = 0$, $\cos\theta = L/(x_2^2 + L^2)^{1/2}$ and $\sin\theta = x_2/(x_2^2 + L^2)^{1/2}$. Thus, by the same means that θ is obtained, ψ is obtained or

$$\psi = \cos^{-1} \frac{\rho}{(x_2^2 + L^2 + z_2^2)^{1/2}} \pm \tan^{-1} \frac{z_2}{(x_2^2 + L^2)^{1/2}} \tag{47}$$

The boundaries of the plate are unknown and, therefore, will be represented by

$$z_2 = f(x_2) \tag{48}$$

integrated from $x_2 = a_1$ to $x_2 = a_2$.

Substituting the limits into Eq. (41) gives

$$A_1F_{12} = \frac{\rho^2}{\pi} \int_{a_1}^{a_2} \int_{f_1(x_2)}^{f_2(x_2)} \int_{\psi_1}^{\psi_2} \int_{\theta_1}^{\theta_2} H\,(x_2,\, z_2,\, \theta,\, \psi)\; d\theta\; d\psi\; dz_2\; dx \tag{49}$$

where $H\,(x_2,\, z_2,\, \theta,\, \psi)$ is the integrand given in Eq. (41), and:

$$\psi_1 = -\cos^{-1} \frac{\rho}{(x_2^2 + L^2 + z_2^2)^{1/2}} + \tan^{-1} \frac{z_2}{(x_2^2 + L^2)^{1/2}}$$

$$\psi_2 = \cos^{-1} \frac{\rho}{(x_2^2 + L^2 + z_2^2)^{1/2}} + \tan^{-1} \frac{z_2}{(x_2^2 + L^2)^{1/2}}$$

$$\theta_1 = -\cos^{-1} \frac{(\rho - z_2 \sin\psi)}{[(x_2^2 + L^2)^{1/2} \cos\psi]} + \tan^{-1} \frac{x_2}{L}$$

$$\theta_2 = \cos^{-1} \frac{(\rho - z_2 \sin\psi)}{[(x_2^2 + L^2)^{1/2} \cos\psi]} + \tan^{-1} \frac{x_2}{L}$$

IV. SPECIFIC SOLUTIONS

Since the technique requires that a separate integral be developed for each combination of geometries, a study of various combinations was undertaken to determine the frequency of occurrence. From the study, the commonly encountered geometries were determined and the integrals were developed. These integrals are presented in the Appendix along with restrictions and individual peculiarities. In those cases where one or more limits of integration are related to the geometry of the surface, such as the surface of a cylinder or a sphere, the limits will be derived. The same caution applies to these integrals as to the previously derived integrals in regard to the selection of the limits. The following integrals will be presented in the simplest form in compliance with complete generality, when possible. It might be added that a further extension of the work could be made by extending the list of integrals and by integrating the existing integrals one or more times, when possible. This latter extension would result in an appreciable decrease in machine computing time.

ACKNOWLEDGEMENT

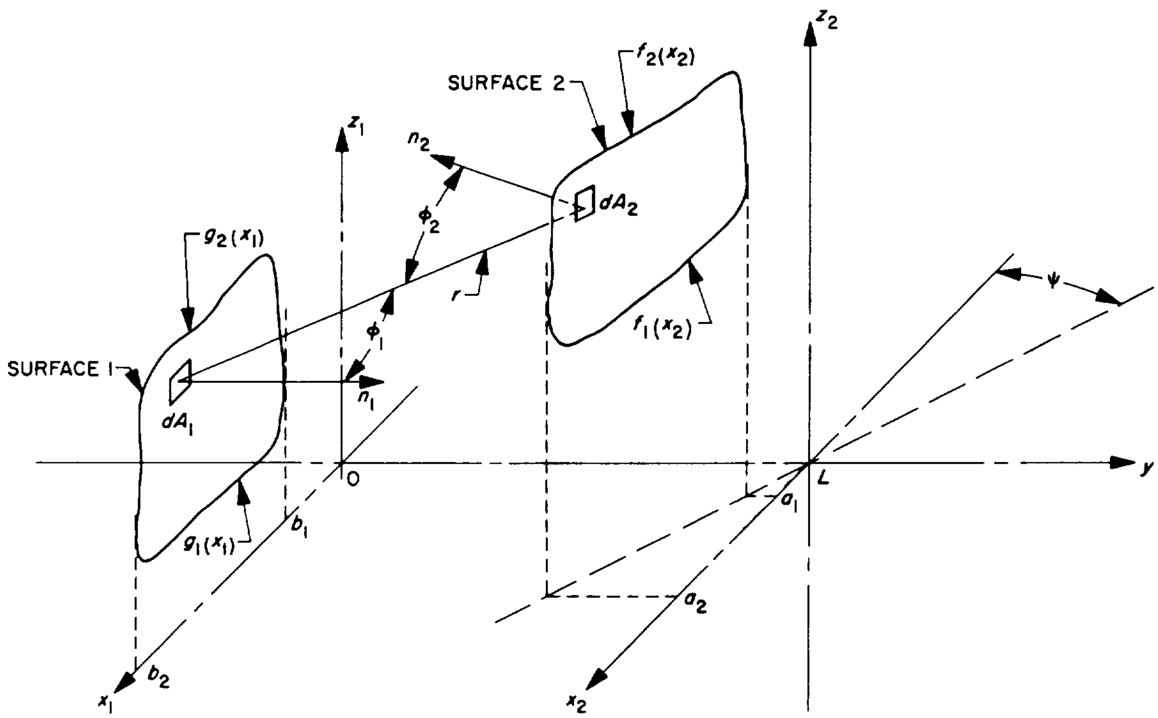
Mrs. Carolyn Level developed many of the integrals given in the Appendix. JPL's Applied Mathematics Group developed computer programs for the integrals. The contributions of these people, together with the useful and thoughtful advice of Dr. John W. Lucas, are gratefully acknowledged.

APPENDIX

Integrals for Configuration Factors for Some Common Geometrical Situations

I. TWO PLATES ARBITRARILY ORIENTED

A. Skewed from Parallel Orientation



$$F_{12} = \frac{1}{\pi A_1} \int_{b_1}^{b_2} \int_{g_1(x_1)}^{g_2(x_1)} \int_{a_1}^{a_2} \int_{f_1(x_2)}^{f_2(x_2)} F(z_2, x_2, z_1, x_1)\, dz_2\, dx_2\, dz_1\, dx_1$$

where

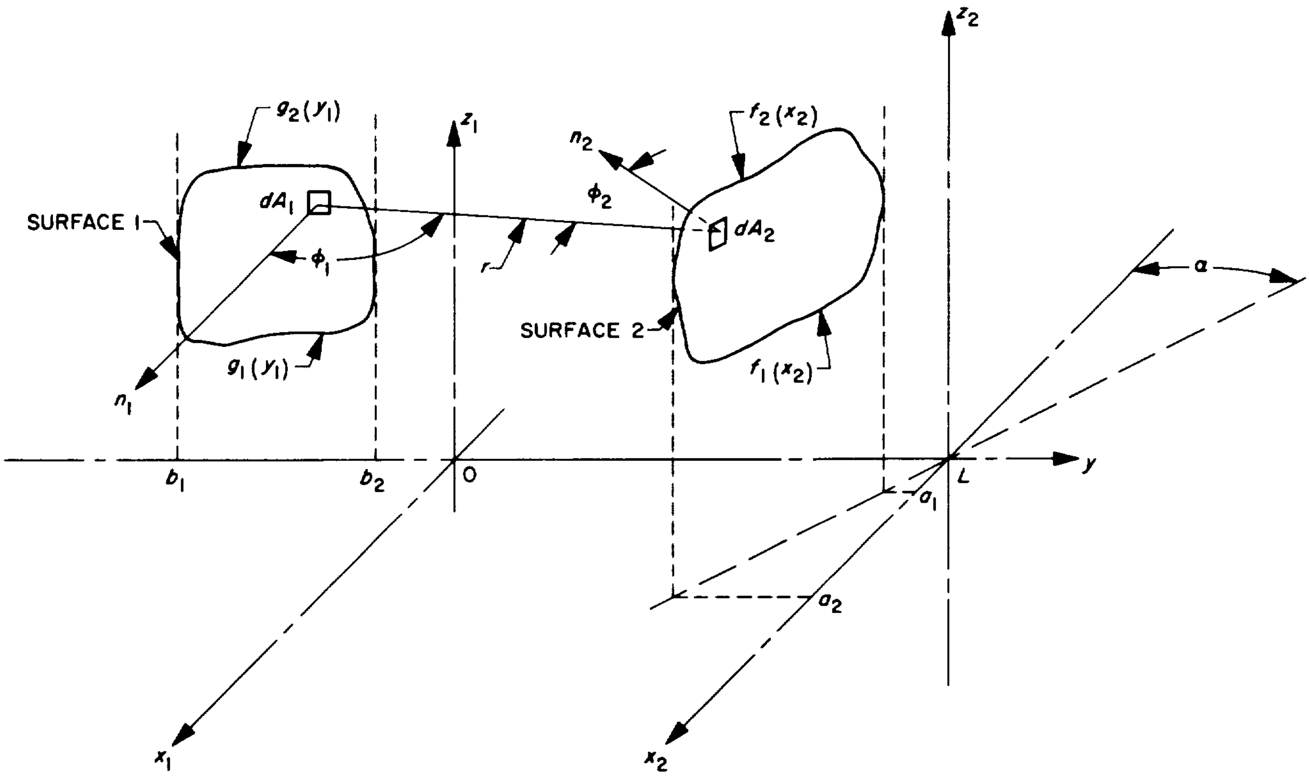
$$F(z_2, x_2, z_1, x_1) = \frac{(L - x_2 \tan \psi)(L - x_1 \tan \psi)}{[(x_2 - x_1)^2 + (L - x_2 \tan \psi)^2 + (z_2 - z_1)^2]^2}$$

Restrictions:

1. $-90 \text{ deg} < \psi < 90 \text{ deg}$
2. Surface 1 must lie in the $y = 0$ plane
3. $f_1(x_2)$ and $f_2(x_2)$ must be written in terms of x_2 and z_2
4. Surface 2 must be located so that it is perpendicular to the $z = 0$ plane

To be supplied: L, ψ, and limits of integration

B. Skewed from Perpendicular Orientation



Restrictions:

1. -90 deg $< \alpha <$ 90 deg
2. Surface 1 must lie in the $x = 0$ plane
3. Surface 2 must be perpendicular to the $z = 0$ plane
4. $f_1(x_2)$ and $f_2(x_2)$ must be written in terms of x_2 and z_2

To be supplied:

L, α, and limits of integration

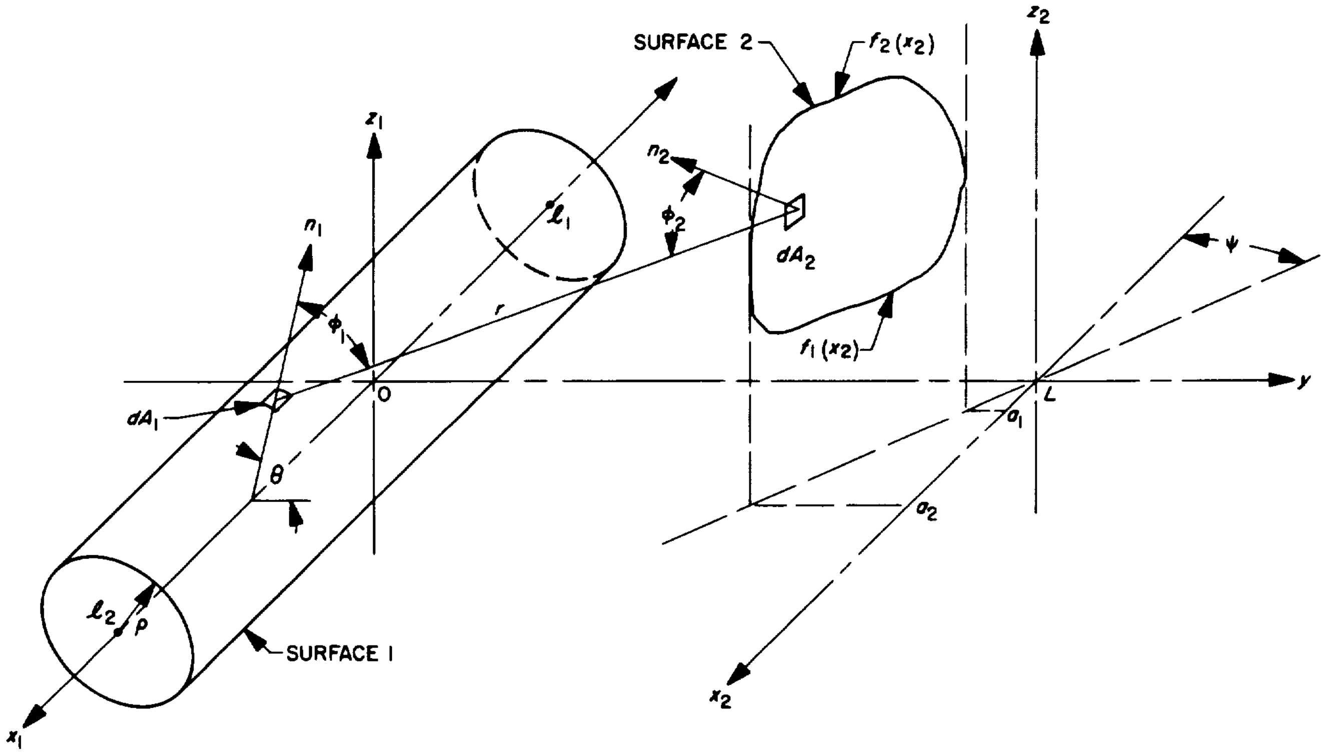
$$F_{12} = \frac{1}{\pi A_1} \int_{b_1}^{b_2} \int_{g_1(y_1)}^{g_2(y_1)} \int_{a_1}^{a_2} \int_{f_1(x_2)}^{f_2(x_2)} F(z_2, x_2, z_1, x_1)\, dz_2\, dx_2\, dz_1\, dy_1$$

where

$$F(z_2, x_2, z_1, x_1) = \frac{x_2 (L - y_1)}{\left[(x_2)^2 + (L - x_2 \tan \alpha - y_1)^2 + (z_2 - z_1)^2\right]^2}$$

II. CYLINDER AND PLATE ARBITRARILY ORIENTED

A. Plate Skewed from Position Parallel to Axis of Cylinder



Restrictions:

1. -90 deg $< \psi <$ 90 deg
2. x_1 axis and cylinder axis must coincide
3. The plate must be perpendicular to the $z = 0$ plane
4. $f_1(x_2)$ and $f_2(x_2)$ must be expressed in terms of x_2 and z_2

To be supplied:

Constants: ρ, L, and ψ

Limits of integration: l_1, l_2, $f_1(x_2)$, $f_2(x_2)$, a_1, and a_2

$$F_{21} = \frac{1}{\pi A_2} \int_{a_1}^{a_2} \int_{f_1(x_2)}^{f_2(x_2)} \int_{l_1}^{l_2} \int_{\theta_1}^{\theta_2} F(\theta, x_1, z_2, x_2)\, d\theta\, dx_1\, dz_2\, dx_2$$

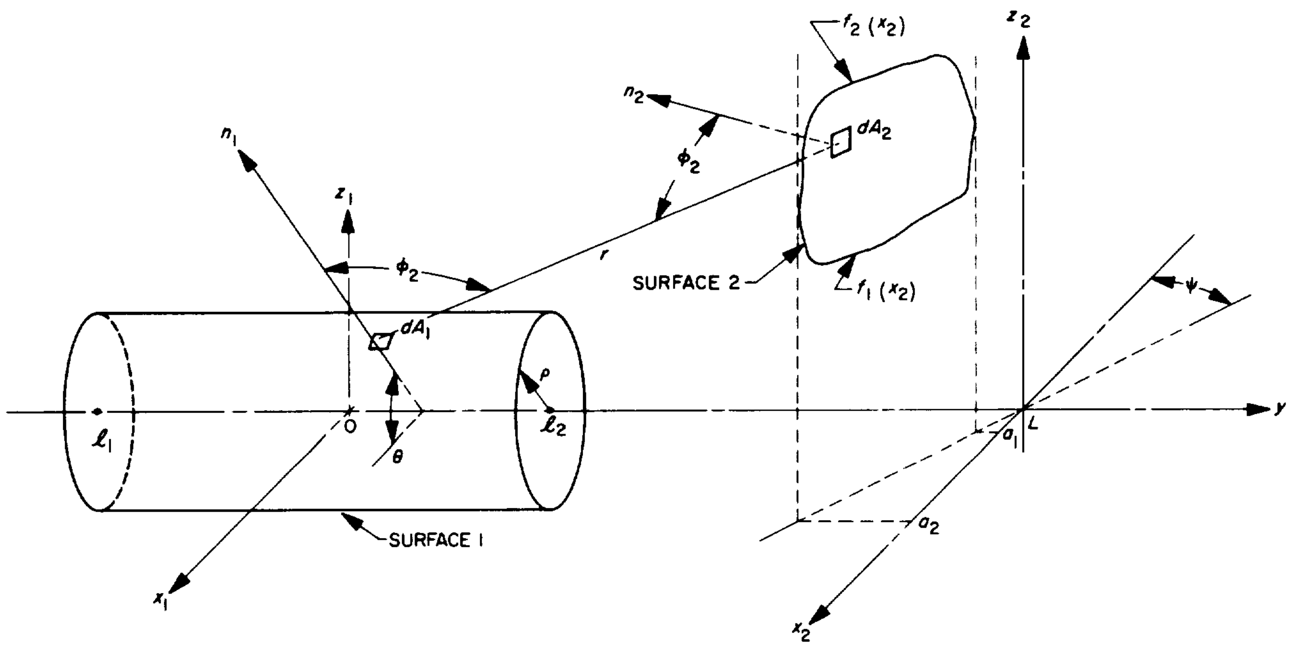
where

$$\theta_1 = -\cos^{-1} \frac{\rho}{[z_2^2 + (L - x_2 \tan\psi)^2]^{1/2}} + \tan^{-1} \frac{z_2}{L - x_2 \tan\psi}$$

$$\theta_2 = \cos^{-1} \frac{\rho}{[z_2^2 + (L - x_2 \tan\psi)^2]^{1/2}} + \tan^{-1} \frac{z_2}{L - x_2 \tan\psi}$$

$$F(\theta, x_1, z_2, x_2) = \frac{\rho\,(L \cos\theta - \rho + z_2 \sin\theta - x_2 \tan\psi \cos\theta)(L - \rho\cos\theta - x_1 \tan\psi)}{[(x_2 - x_1)^2 + (L - x_2 \tan\psi - \rho\cos\theta^2 + (z_2 - \rho \sin\theta)^2]^2}$$

B. Plate Skewed from Position Perpendicular to Axis of Cylinder



Restrictions:

1. -90 deg $< \psi < 90$ deg
2. y axis and cylinder axis must coincide
3. Plate must be perpendicular to $z_2 = 0$ plane
4. $f_1(x_2)$ and $f_2(x_2)$ must be expressed in terms of x_2 and z_2 only
5. Any portion of plate which lies within the circle $x_2^2 + z_2^2 = \rho^2$ must be excluded in the integration

To be supplied:

Constants: L, ρ, and ψ

Limits of integration: a_1, a_2, $f_1(x_2)$, $f_2(x_2)$, l_1, and l_2

$$F_{21} = \frac{\rho}{\pi A_2} \int_{a_1}^{a_2} \int_{f_1(x_2)}^{f_2(x_2)} \int_{l_1}^{l_2} \int_{\theta_1}^{\theta_2} F(\theta, y, z_2, x_2)\, d\theta\, dy\, dz_2\, dx_2$$

where

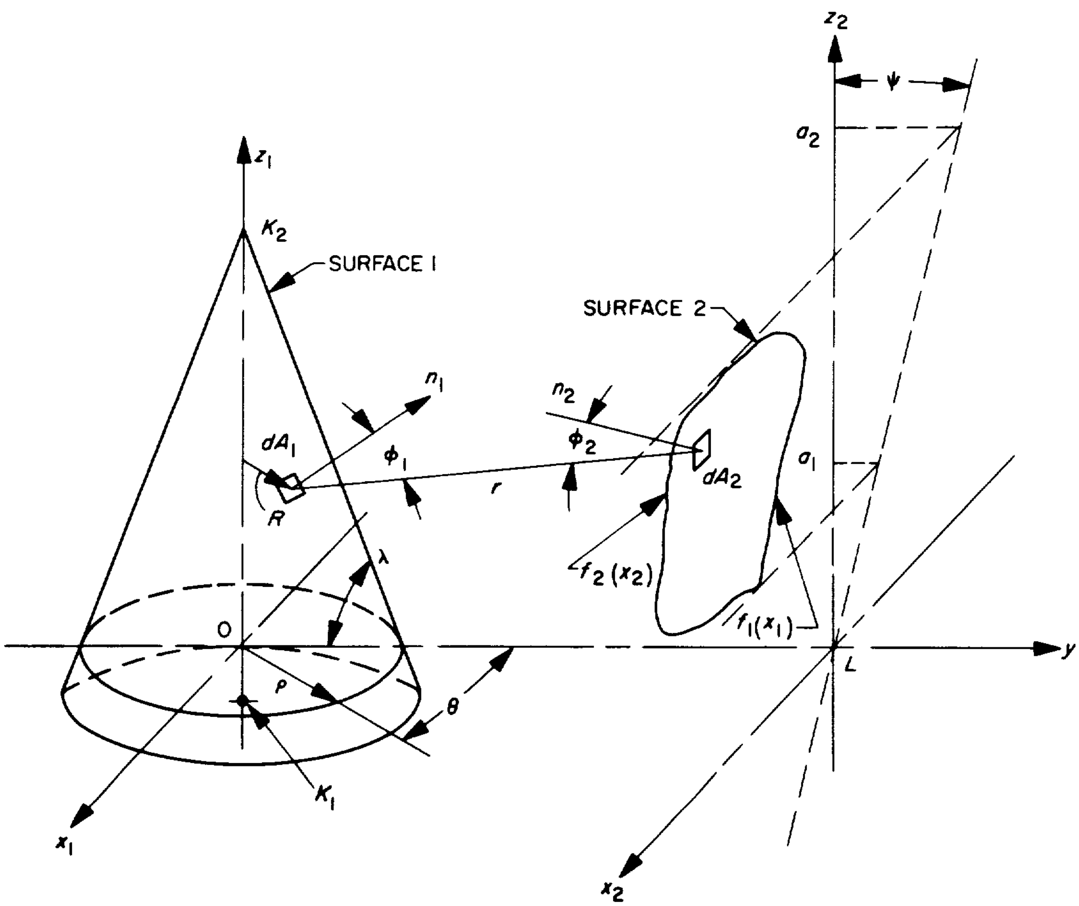
$$\theta_1 = -\cos^{-1} \frac{\rho}{(x_2^2 + z_2^2)^{1/2}} + \tan^{-1} \frac{z_2}{x_2}$$

$$\theta_2 = \cos^{-1} \frac{\rho}{(x_2^2 + z_2^2)^{1/2}} + \tan^{-1} \frac{z_2}{x_2}$$

$$F(\theta, y, z_2, x_2) = \frac{(x_2 \cos\theta - \rho + z_2 \sin\theta)(L - y - \rho\cos\theta\tan\psi)}{[(\rho\cos\theta - x_2)^2 + (y - L + x_2\tan\psi)^2 + (\rho\sin\theta - z_2)^2]^2}$$

III. CONE AND PLATE ARBITRARILY SKEWED

A. Plate Skewed from Position Parallel to Axis of Cone



Restrictions:

1. -90 deg $< \psi <$ 90 deg
2. Axis of Cone must coincide with z_1 axis
3. Plate must be positioned perpendicular to $x = 0$ plane
4. $f_1(x_2)$ and $f_2(x_2)$ must be expressed in terms of x_2 and z_2
5. ρ must be radius of cone in the $z = 0$ plane

To be supplied:

Constants: L, ρ, K_2, λ, and ψ

Limits of integration: K_1, K_2, $f_1(x_2)$, $f_2(x_2)$, a_1, and a_2

$$F_{21} = \frac{1}{\pi A_2} \int_{a_1}^{a_2} \int_{f_1(x_2)}^{f_2(x_2)} \int_{K_1}^{K_3} \int_{\theta_1}^{\theta_2} F(\theta, z_1, z_2, x_2)\, d\theta\, dz_1\, dx_2\, dz_2$$

where

$$\theta_1 = -\cos^{-1} \frac{R + (z_1 - z_2) \cot \lambda}{[(L + z_2 \tan \psi)^2 + x_2^2)^{1/2}} + \tan^{-1} \frac{x_2}{L + z_2 \tan \psi}$$

$$\theta_2 = \cos^{-1} \frac{R + (z_1 - z_2) \cot \lambda}{[(L + z_2 \tan \psi)^2 + x_2^2]^{1/2}} + \tan^{-1} \frac{x_2}{L + z_2 \tan \psi}$$

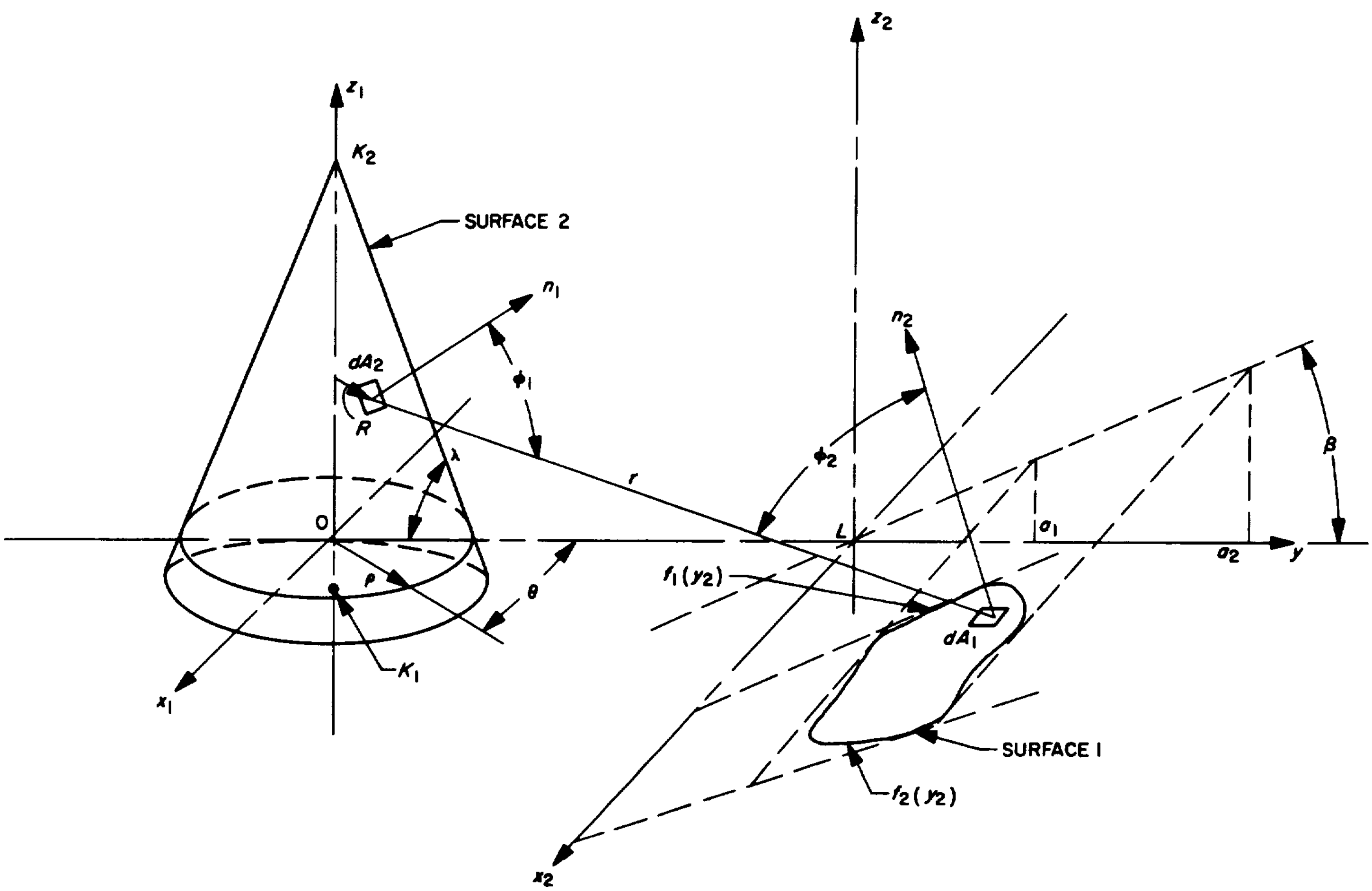
$$F(\theta, z_1, z_2, x_2) = \frac{R\,[(L + z_2 \tan \psi) \cos \theta + x_2 \sin \theta + (z_2 - z_1) \cot \lambda - R]\,[L + z_1 \tan \psi - R \cos \theta]}{[(x_2 - R \sin \theta)^2 + (L + z_2 \tan \psi - R \cos \theta)^2 + (z_2 - z_1)^2]^2}$$

and R is derived as follows:

$$R = \rho - \frac{z}{\tan \lambda} \qquad \text{where} \qquad \tan \lambda = \frac{K_2}{\rho}, \quad K_2 = \text{cone apex location}$$

therefore $$R = \rho \left(1 - \frac{z}{K_2}\right)$$

B. Plate Skewed from Position Perpendicular to Axis of Cone



Restrictions:

1. -90 deg $< \beta <$ 90 deg
2. Axis of cone must coincide with z_1 axis
3. Plate must be positioned perpendicular to $x = 0$ plane
4. $f_1(y_2)$ and $f_2(y_2)$ must be expressed in terms of x_2 and y_2
5. ρ must be radius of cone in the $z = 0$ plane

To be supplied:

Constants: L, ρ, K_2, λ, and β

Limits of integration: K_1, K_2, $f_1(y_2)$, $f_2(y_2)$, a_1, and a_2

$$A_1F_{12} = \frac{1}{\pi} \int_{a_1}^{a_2} \int_{f_1(y_2)}^{f_2(y_2)} \int_{K_1}^{K_3} \int_{\theta_1}^{\theta_2} F(\theta, z_1, x_2, y_2)\, d\theta\, dz_1\, dx_2\, dy_2$$

where

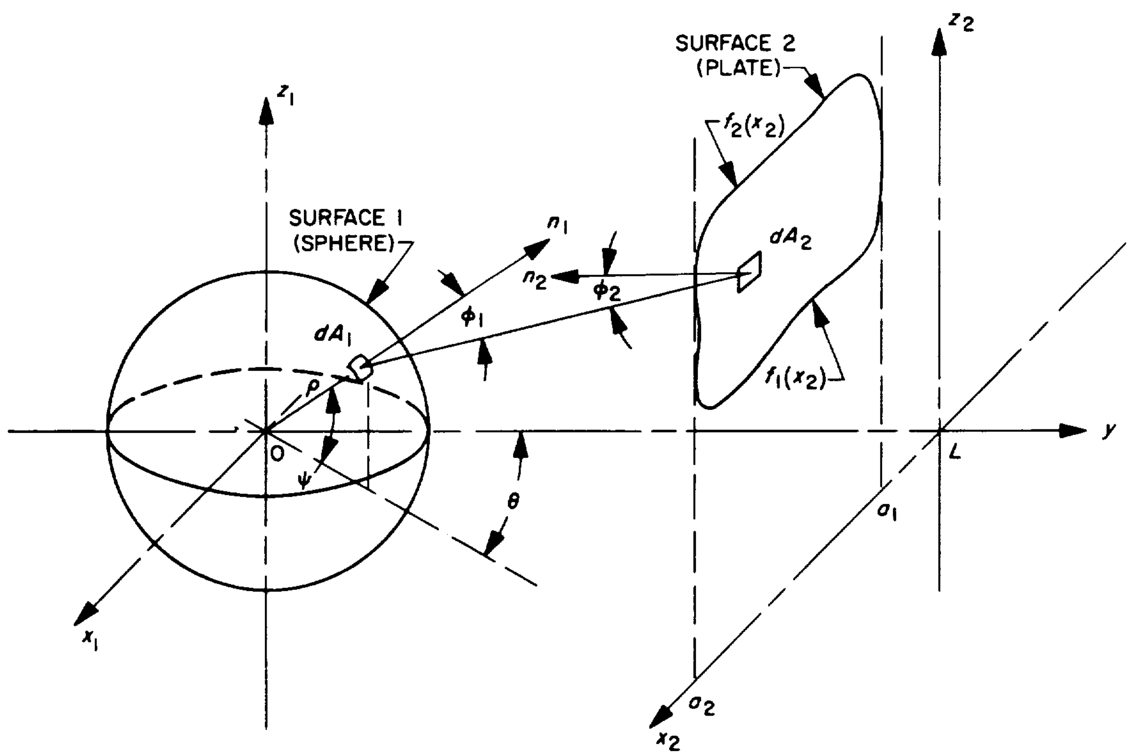
$$\theta_1 = -\cos^{-1} \frac{R - \cot\lambda\,(y_2 \tan\beta - z_1)}{[(L + y_2)^2 + x_2^2]} + \tan^{-1} \frac{x_2}{L + y_2}$$

$$\theta_2 = \cos^{-1} \frac{R - \cot\lambda\,(y_2 \tan\beta - z_1)}{[(L + y_2)^2 + x_2^2]^{1/2}} + \tan^{-1} \frac{x^2}{L + y^2}$$

$$F(\theta, z_1, x_2, y_2) = \frac{R\,[(L + y_2 \cos\theta - R + x_2 \sin\theta + \cot\lambda\,(y_2 \tan\beta - z_1)]\,[(L - R\cos\theta)\tan\beta + z_1]}{[(x_2 - R\sin\theta)^2 + (L + y_2 - R\cos\theta)^2 + (z - y_2 \tan\beta)^2]^2}$$

and $y_2 = y - L$, $\tan\lambda = K_2/\rho$, $R = \rho(1 - z_1/K_2)$, and K_2 is the location of the cone apex.

IV. SPHERE AND PLATE



Restrictions:

1. Sphere must be located at $x_1 = 0$; $y_1 = 0$; $z_1 = 0$
2. Plate must be perpendicular to both $x = 0$ and $z = 0$ planes

To be supplied:

Constants: L and ρ

Limits of integration: a_1, a_2, $f_1(x_2)$, and $f_2(x_2)$

$$F_{21} = \frac{\rho^2}{\pi A_2} \int_{a_1}^{a_2} \int_{f_1(x_2)}^{f_2(x_2)} \int_{\psi_1}^{\psi_2} \int_{\theta_1}^{\theta_2} F(\theta, \psi, z_2, x_2)\, d\theta\, d\psi\, dz_2\, dx_2$$

where

$$\psi_1 = -\cos^{-1} \frac{\rho}{(x_2^2 + L^2 + z_2^2)^{1/2}} + \tan^{-1} \frac{z_2}{(x_2^2 + L^2)^{1/2}}$$

$$\psi_2 = \cos^{-1} \frac{\rho}{(x_2^2 + L^2 + z_2^2)^{1/2}} + \tan^{-1} \frac{z_2}{(x_2^2 + L^2)^{1/2}}$$

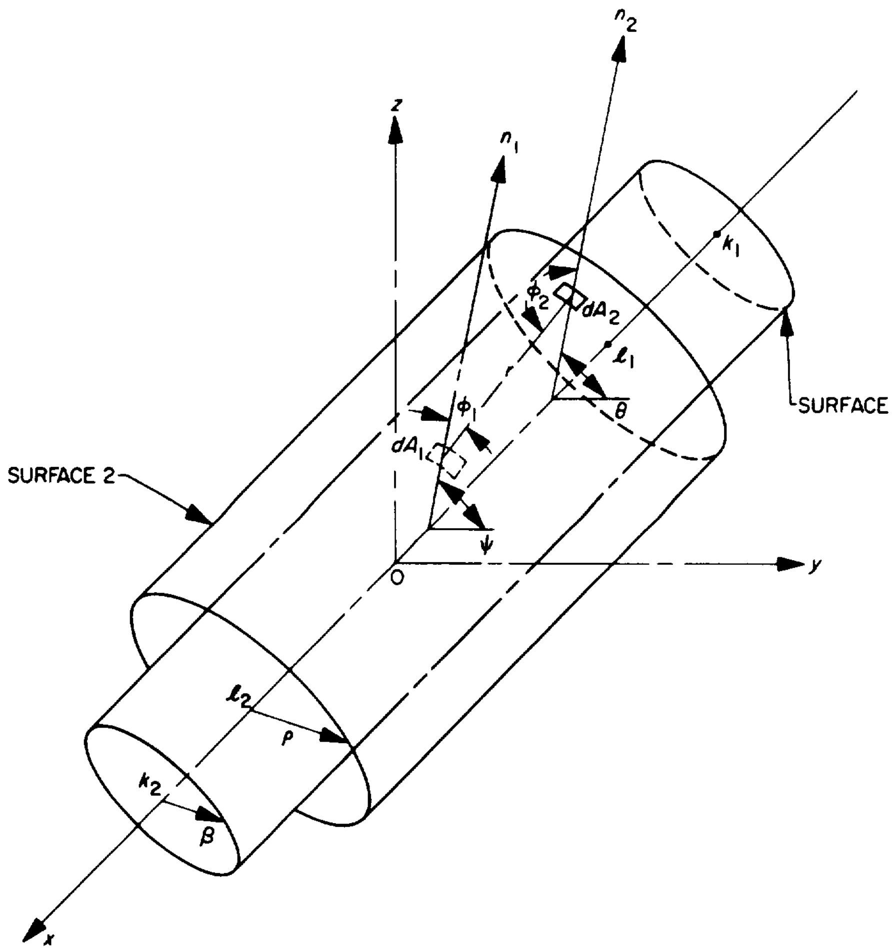
$$\theta_1 = -\cos^{-1} \frac{\rho - z_2 \sin \psi}{(x_2^2 + L^2)^{1/2} \cos \psi} + \tan^{-1} \frac{x_2}{L}$$

$$\theta_2 = \cos^{-1} \frac{\rho - z_2 \sin \psi}{(x_2^2 + L^2)^{1/2} \cos \psi} + \tan^{-1} \frac{x_2}{L}$$

$$F(\theta, \psi, z_2, x_2) = \frac{[(x_2 \sin \theta + L \cos \theta) \cos \psi + z_2 \sin \psi - \rho](L - \rho \cos \psi \cos \theta)}{[(x_2 - \rho \cos \psi \sin \theta)^2 + (L - \rho \cos \psi \cos \theta)^2 + (z_2 - \rho \sin \psi)^2]^2}$$

V. CYLINDER TO CYLINDER (TWO SPECIAL CASES)

A. Concentric Cylinders



Restriction:

1. Cylinders must be parallel and concentric

To be provided:

Constants: β and ρ

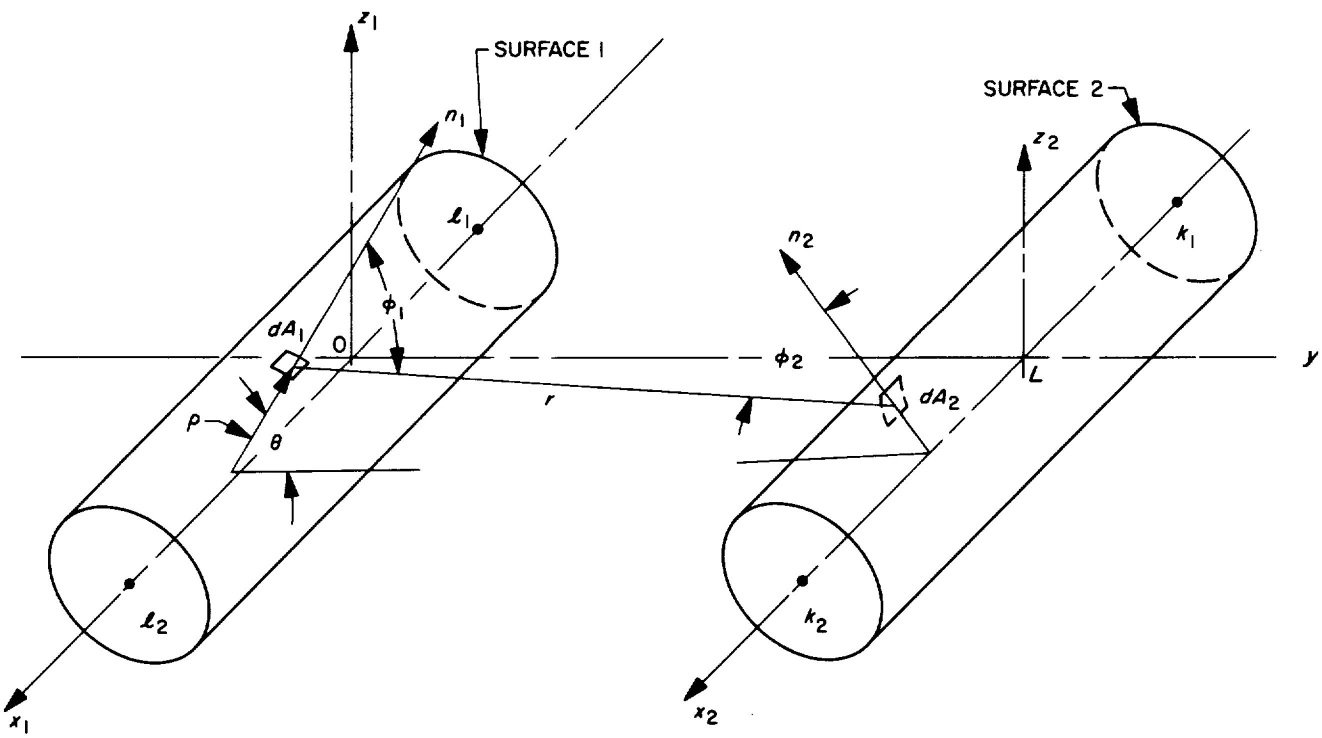
Limits of integration: l_1, l_2, θ_1, θ_2, k_1, and k_2

$$A_1F_{12} = \frac{\beta\rho}{\pi} \int_{l_1}^{l_2} \int_{\theta_1}^{\theta_2} \int_{k_1}^{k_2} \int_{\psi_1}^{\psi_2} F(\theta, \psi, x_1, x_2)\, d\psi\, dx_1\, d\theta\, dx_2$$

where $\psi_1 = -\cos^{-1} \beta/\rho + \theta$, $\psi_2 = \cos^{-1} \beta/\rho + \theta$, and

$$F(\theta, \psi, x_1, x_2) = \frac{[(\rho \cos(\theta - \psi) - \beta]\,[\beta \cos(\theta - \psi) - \rho]}{[(x_1 - x_2)^2 + (\rho \cos\theta - \beta \cos\psi)^2 + (\rho \sin\theta - \beta \sin\psi)^2]^2}$$

B. Parallel Cylinder



Restrictions:

1. $\rho + \zeta < L$
2. Cylinders must be parallel
3. The axis of one cylinder must coincide with x_1 axis

To be supplied:

Constants ρ, ζ (radii of cylinders), and L

Limits of integration: l_1, l_2, k_1, and k_2

$$A_1F_{12} = \int_{l_1}^{l_2} \int_{k_1}^{k_2} \int_{\cos^{-1}(\rho+\zeta)/L}^{\cos^{-1}(\zeta-\rho)/L} \int_{\theta_1}^{\theta_2} F(x_1, x_2, \psi, \theta)\, d\theta\, d\psi\, dx_2\, dx_1$$

$$+ \int_{l_1}^{l_2} \int_{k_1}^{k_2} \int_{-\cos^{-1}(\zeta-\rho)/L}^{-\cos^{-1}(\rho+\zeta)/L} \int_{\theta_3}^{\theta_4} F(x_1, x_2, \psi, \theta)\, d\theta\, d\psi\, dx_2\, dx_1$$

$$+ \int_{l_1}^{l_2} \int_{k_1}^{k_2} \int_{-\cos^{-1}(\rho+\zeta)/L}^{\cos^{-1}(\rho+\zeta)/L} \int_{\theta_2}^{\theta_3} F(x_1, x_2, \psi, \theta)\, d\theta\, d\psi\, dx_2\, dx_1$$

where

$$F(x_1, x_2, \psi\, \theta) = \frac{\zeta\rho}{\pi} \frac{[L\cos\theta - \rho - \zeta\cos(\theta+\psi)]\,[L\cos\psi - \zeta - \rho\cos(\theta+\psi)]}{[(x_1-x_2)^2 + (L-\rho\cos\theta-\zeta\cos\psi)^2 + (\rho\sin\theta - \zeta\sin\psi)^2]^2}$$

$$\theta_1 = -\cos^{-1}\frac{\rho - L\cos\psi}{\zeta} + \psi$$

$$\theta_2 = -\cos^{-1}\frac{\zeta}{(\rho^2 + L^2 - 2L\rho\cos\psi)^{1/2}} + \tan^{-1}\frac{\rho\sin\psi}{L-\rho\cos\psi}$$

$$\theta_3 = \cos^{-1}\frac{\zeta}{(\rho^2 + L^2 - 2L\rho\cos\psi)^{1/2}} + \tan^{-1}\frac{\rho\sin\psi}{L-\rho\cos\psi}$$

$$\theta_4 = \cos^{-1}\frac{\rho - L\cos\psi}{\zeta} - \psi$$